Quantum mechanics and operator algebras on the Hilbert ball (The revised)

Katsunori Kawamura*

College of Science and Engineering Ritsumeikan University,
1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577, Japan

Abstract

Cirelli, Manià and Pizzocchero generalized quantum mechanics by Kähler geometry. Furthermore they proved that any unital C*-algebra is represented as a function algebra on the set of pure states with a noncommutative $*$-product as an application. The ordinary quantum mechanics is regarded as a dynamical system of the projective Hilbert space $\mathcal{P}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$. The space $\mathcal{P}(\mathcal{H})$ is an infinite dimensional Kähler manifold of positive constant holomorphic sectional curvature. In general, such dynamical system is constructed for a general Kähler manifold of nonzero constant holomorphic sectional curvature c. The Hilbert ball $B_{\mathcal{H}}$ is defined by the open unit ball in $\mathcal{H}$ and it is a Kähler manifold with $c < 0$. We introduce the quantum mechanics on $B_{\mathcal{H}}$. As an application, we show the structure of the noncommutative function algebra on $B_{\mathcal{H}}$.

MSC: 81R15; 32Q15; 58B20; 46L65

Keywords: Hilbert ball; trial quantum mechanics; Kähler geometry; operator algebra

1 Introduction

We study a generalization of quantum mechanics by using the theory of operator algebra and the Kähler geometry. The geometric quantization [25] is well-known as a geometrical approach of quantization. In this theory, the

*e-mail: kawamura@kurims.kyoto-u.ac.jp.

geometry means the symplectic structure of the phase space associated with the classical mechanics. On the other hand, our interest is a geometry of the quantum mechanics itself without considering the classical mechanics and the phase space. Hence we do not treat the problem of quantization in this paper.

A trial quantum system was considered by Cirelli, Manià and Pizzocchero in order to study the quantum mechanics from a standpoint of geometry [8]. The state space in a trial quantum system is a complete, connected, simply connected Hilbert Kähler manifold of constant holomorphic sectional curvature c. If $c > 0$, then the system is the ordinary quantum system with the projective Hilbert space as the state space (see also [13]). Furthermore they showed that a unital C*-algebra $\mathfrak{A}$ is represented by Kähler functions on the Kähler bundle which is constructed by the set of pure states and the spectrum of $\mathfrak{A}$ in [9]. This result is an application of the case $c > 0$.

We examine *what happens when* $c < 0$. The study of the case $c < 0$ means both a thought experiment of a new quantum mechanics and a new theory of operator algebra from a standpoint of geometry. In stead of the projective Hilbert space, we consider a framework of quantum mechanics on the Hilbert ball

$$B_{\mathcal{H}} \equiv \{z \in \mathcal{H} : \|z\| < 1\} \tag{1.1}$$

for a complex Hilbert space $\mathcal{H}$ as a trial quantum system. The space $B_{\mathcal{H}}$ is an example of Kähler manifold of negative constant holomorphic sectional curvature. We describe its quantum mechanics by calculating objects appearing in its mechanics, that is,

state space of system	$B_{\mathcal{H}}$
observable	Kähler function on $B_{\mathcal{H}}$
symmetry	isometry of $B_{\mathcal{H}}$
transition probability	distance of $B_{\mathcal{H}}$

We describe the Kähler algebra on $B_{\mathcal{H}}$ as an operator algebra by solving equations of definition of Kähler functions on $B_{\mathcal{H}}$.

In this section, we provide rough overviews for each subject: Kähler geometry, geometrical quantum mechanics, Hilbert Kähler manifold $B_{\mathcal{H}}$, operator algebra, their relations and main results.

1.1 Kähler geometry

We briefly review the definition of Kähler manifold [16]. In this paper, any (infinite dimensional) manifold means a Hilbert manifold with respect to

the Fréchet derivative [1, 6, 7, 15]. Let M be a complex manifold with the almost complex structure J. If a Riemannian metric g on M satisfies $g(Jx, Jx) = g(x, y)$, then g is called a *Hermitian metric* on M. For a Hermitian metric g, the 2-differential form ω defined by

$$\omega(x, y) = g(Jx, y) \tag{1.2}$$

is called the *Kähler form* of M. If $d\omega = 0$, then M is called a *Kähler manifold* with the *Kähler metric* g. Let R be the curvature tensor of the metric connection of M. A Kähler manifold M is said to be *of constant holomorphic sectional curvature* $c \in \mathbf{R}$ if $g_p(U_p, R_p(V_p, U_p, V_p)) = c$ for every $p \in M$ and any orthonormal basis U_p, V_p of every J_p-invariant two-plane of T_pM. Let $C^\infty(M)$ be the set of all smooth functions on M. For $f \in C^\infty(M)$, the *Hamiltonian vector field* Idf [3] of f is defined by

$$\omega(Idf, Y) = df(Y) \quad (Y \in TM).$$

The *Poisson bracket* $\{\cdot, \cdot\}$ of M is defined by

$$\{f, l\} \equiv \omega(Idf, Idl) \quad (f, l \in C^\infty(M)).$$

The tangent space of M is a Hilbert space with respect to the inner product associated with the Kähler metric. We use the complexified tangent space

$$\mathcal{T}_pM \equiv T_pM \oplus \overline{T_pM}$$

where $\overline{T_pM}$ is the conjugate Hilbert space of T_pM. For $f \in C^\infty(M)$, the holomorphic vector fields $\mathrm{grad} f$ and $\mathrm{sgrad} f$ are defined by

$$g(\mathrm{grad} f, \bar{Y}) = \bar{\partial} f(\bar{Y}), \qquad \omega(\mathrm{sgrad} f, \bar{Y}) = \bar{\partial} f(\bar{Y}) \tag{1.3}$$

for any antiholomorphic vector field $\bar{Y}$ on M where $\bar{\partial} f$ is the antiholomorphic differential of f. We call $\mathrm{grad} f$ and $\mathrm{sgrad} f$ by the *holomorphic gradient* and the *holomorphic skew-gradient* of f, respectively. The differential df of a function f is written by the holomorphic part and antiholomorphic part $\partial f + \bar{\partial} f$. Then $Idf = \mathrm{sgrad} f + \overline{\mathrm{sgrad} f}$ and $\mathrm{grad} f = J\mathrm{sgrad} f$.

By the Kuiper's theorem [19], the tangent bundle of any infinite dimensional Hilbert manifold is a trivial bundle. Furthermore if two infinite dimensional Hilbert manifolds are homotopy equivalent, then they are diffeomorphic. However, the Kähler structure of them are different in general. In fact, we treat two infinite dimensional Hilbert Kähler manifolds of positive and negative holomorphic sectional curvatures, respectively.

1.2 Reformulation of quantum mechanics by Kähler geometry

We review the foundation of quantum mechanics [20, 23]. In quantum mechanics, a state is represented by a non zero vector in a complex Hilbert space $\mathcal{H}$. Two vectors v and w are identified as a state when there exits $c \in \mathbf{C}$ such that $w = cv$. The equivalence class of non zero vectors in $\mathcal{H}$ by this identification is called a *unit ray* [25]. Hence states of the system are represented as the set of unit rays of $\mathcal{H}$. Therefore the set of all states is the *projective Hilbert space*

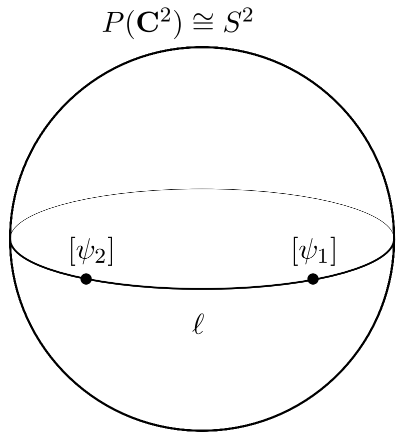
$$\mathcal{P}(\mathcal{H}) \equiv (\mathcal{H} \setminus \{0\})/\mathbf{C}^{\times} \tag{1.4}$$

of $\mathcal{H}$ where $\mathbf{C}^{\times} \equiv \{z \in \mathbf{C} : z \neq 0\}$. It is known that $\mathcal{P}(\mathcal{H})$ is a Kähler manifold as a Hilbert manifold for $\mathcal{H}$ with any dimension.

In [8], Cirelli, Manià and Pizzocchero introduced a geometric framework of quantum mechanics. We briefly review it here. The operator theory of quantum mechanics is reformulated by the Kähler geometry of $\mathcal{P}(\mathcal{H})$. The Schrödinger equation is written by the equation of flow of a Hamiltonian vector field on $\mathcal{P}(\mathcal{H})$. The transition probability of two states is the distance of two points in $\mathcal{P}(\mathcal{H})$ with respect to the Kähler metric. Observables are functions on $\mathcal{P}(\mathcal{H})$ and the product among operators is rewritten by the $*$-product among functions associated with the Kähler form. A symmetry of the system is given by an isometry of $\mathcal{P}(\mathcal{H})$ with respect to the Kähler metric.

physics	**operator theory**	**Kähler geometry**
state space of system	set of unit rays of $\mathcal{H}$	$\mathcal{P}(\mathcal{H})$
observables	linear operators	Kähler functions on $\mathcal{P}(\mathcal{H})$
symmetry	unitary operator	isometry of $\mathcal{P}(\mathcal{H})$
transition probability	absolute value of inner product	Kähler distance of $\mathcal{P}(\mathcal{H})$
equation of motion	Schrödinger equation	flow equation of Hamiltonian vector field on $\mathcal{P}(\mathcal{H})$

For example, if $\dim \mathcal{H} = 2$, then we can identify $\mathcal{P}(\mathcal{H})(= P(\mathbf{C}^2))$ with the Riemann sphere S^2 as a Riemannian manifold by the stereographic projection. Two state vectors $\psi_1, \psi_2 \in \mathcal{H}$ are corresponded to two points $[\psi_1], [\psi_2] \in S^2$. The geodesic ℓ through $[\psi_1]$ and $[\psi_2]$ in S^2 is always a maximal circle in S^2.



The Kähler distance $d([\psi_1],[\psi_2])$ between $[\psi_1]$ and $[\psi_2]$ is defined by the length of the geodesic between them and the following holds:

$$d([\psi_1],[\psi_2]) = \sqrt{2}\arccos|\langle\psi_1|\psi_2\rangle|$$

where $\langle\psi_1|\psi_2\rangle$ is the inner product of ψ_1 and ψ_2 and we assume that $\|\psi_1\| = \|\psi_2\| = 1$.

1.3 Generalization of quantum mechanics by Kähler geometry admitting negative Planck constant

Cirelli, Manià and Pizzocchero generalized the correspondence in § 1.2 to a general Hilbert manifold with some requirements and concluded that Kähler manifolds are necessary for physical ingredients as state manifolds [8]. We review the theory without the original assumption for the positivity of the Plank constant $\hbar$.

Definition 1.1 *A data $(M,\omega,g,\{\Phi_t\}_{t\in\mathbf{R}})$ is a trial quantum system if the following is satisfied:*

(i) *M is a real smooth Hilbert manifold with a symplectic form ω and a Riemannian metric g.*

(ii) *$\{\Phi_t\}_{t\in\mathbf{R}}$ is a continuous one parameter group of smooth mappings on M such that Φ_t preserves both ω and g, that is, $\Phi_t^*\omega = \omega$ and $\Phi_t^* g = g$ for each $t \in \mathbf{R}$.*

The physical meaning of them is as follows. The manifold M is the set of pure states of a system. The set

$$\mathcal{K}(M,\mathbf{R}) \equiv \{f \in C^\infty(M,\mathbf{R}); L_{Idf}\,g = 0\} \tag{1.5}$$

is the set of observables where L_{Idf} is the Lie derivative by Idf. The symplectic form ω gives a Hamiltonian system as same as classical mechanics. The Riemannian metric g gives the dispersion structure

$$\Delta f \equiv \sqrt{\frac{|\hbar|}{2} g(Idf, Idf)} \tag{1.6}$$

of an observable f where the real number $\hbar$ is the Plank constant in the system. The family $\{\Phi_t\}_{t\in\mathbf{R}}$ gives a dynamical law in the system.

Definition 1.2 *Let $(M, \omega, g, \{\Phi_t\}_{t\in\mathbf{R}})$ be a trial quantum system.*

(i) *A symmetry of $(M, \omega, g, \{\Phi_t\}_{t\in\mathbf{R}})$ is a smooth map which preserves both ω and g.*

(ii) *The transition probability between p and q in M is the distance between p and q with respect to g.*

If the distance of M is not bounded, then we can not regard the distance as a probability.

We call (M, ω, g) the *state manifold* of $(M, \omega, g, \{\Phi_t\}_{t\in\mathbf{R}})$. Let T_pM and T_p^*M be the tangent space and the cotangent space of M at $p \in M$, respectively.

Definition 1.3 (i) *The set $\mathcal{K}(M, \mathbf{R})$ in (1.5) is full if the linear span of $\{d_pf : f \in \mathcal{K}(M, \mathbf{R})\}$ equals to T_p^*M for any $p \in M$.*

(ii) *The manifold M satisfies the uncertainty principle if M satisfies the following conditions for Δf in (1.6):*

(a) $\Delta_p f \cdot \Delta_p l \geq |\frac{\hbar}{2}\{f, l\}_p|$ *for any $f, l \in \mathcal{K}(M, \mathbf{R})$ and $p \in M$ where $\{\cdot, \cdot\}$ is the Poisson bracket with respect to ω.*

(b) $\Delta_p f = \inf\{\lambda \geq 0 : \lambda \cdot \Delta_p l \geq |\frac{\hbar}{2}\{f, l\}_p|$, *for every $l \in \mathcal{K}(M, \mathbf{R})$*$\}$ *for every $f \in \mathcal{K}(M, \mathbf{R})$ and $p \in M$.*

For $p \in M$, define the linear map J_p from T_pM to T_pM by $\omega_p(u, v) = g_p(J_pu, v)$ for $u, v \in T_p(M)$. For $\nu \in \mathbf{R} \setminus \{0\}$, define

$$f *_\nu l \equiv f \cdot l + \frac{1}{2}\nu \left(g(Idf, Idl) + \sqrt{-1}\omega(Idf, Idl) \right) \quad (f, l \in \mathcal{K}(M, \mathbf{C})) \tag{1.7}$$

where $\mathcal{K}(M, \mathbf{C}) \equiv \{f : M \to \mathbf{C}; \mathrm{Re}f, \mathrm{Im}f \in \mathcal{K}(M, \mathbf{R})\}$. The set $\mathcal{K}(M, \mathbf{C})$ is not closed with respect to $*_\nu$ in general. But we call it the $*$-*product* of $\mathcal{K}(M, \mathbf{C})$ from here. By (1.7),

$$f *_\nu l - l *_\nu f = \sqrt{-1}\nu\{f, l\}. \tag{1.8}$$

Proposition 1.4 *Let M be a state manifold.*

(i) *If $\mathcal{K}(M, \mathbf{R})$ is full, then M satisfies the uncertainty principle if and only if $J^2 = -1$.*

(ii) *If J is integrable, that is, $\nabla J = 0$, then for any $\nu \in \mathbf{R} \setminus \{0\}$,*

$$f *_\nu (l *_\nu m) - (f *_\nu l) *_\nu m = 0 \quad (f, l, m \in \mathcal{K}(M, \mathbf{C})). \tag{1.9}$$

(iii) *If $\mathcal{K}(M, \mathbf{R})$ is full and (1.9) holds for any $f, l, m \in \mathcal{K}(M, \mathbf{C})$, then J is integrable.*

Proof. For (i), see Proposition 4.5 in [8], p 2896. For (ii) and (iii), see Proposition 4.6 in [8], p 2897. □

From here, we assume that (M, J, g) is a Kähler manifold. Then a symmetry of (M, J, g) is a Kähler (generally not holomorphic) isometry on (M, J, g) and (1.7) is rewritten as follows:

$$f *_\nu l = f \cdot l + \nu \cdot \partial f(\mathrm{grad} l). \tag{1.10}$$

Proposition 1.5 (i) *If M is a Kähler manifold of constant holomorphic sectional curvature $2/\nu$, then $\mathcal{K}(M, \mathbf{C})$ is closed with respect to the $*$-product $*_\nu$ in (1.10) and $*_\nu$ is associative.*

(ii) *If $\mathcal{K}(M, \mathbf{R})$ is full and $\mathcal{K}(M, \mathbf{C})$ is closed with respect to the the $*$-product $*_\nu$, then M has the constant holomorphic sectional curvature $2/\nu$.*

Proof. These are shown in Proposition 4.3 in the part II of [8]. □
From Proposition 1.5, if (M, J, g) is a Kähler manifold of non zero constant holomorphic sectional curvature c, then $\mathcal{K}(M, \mathbf{C})$ is closed with respect to the product $*$ defined by

$$f * l \equiv f \cdot l + \frac{2}{c} \cdot \partial f(\mathrm{grad} l) \quad (f, l \in \mathcal{K}(M, \mathbf{C})). \tag{1.11}$$

From (1.8) and (1.11), we obtain the following relation between the holomorphic sectional curvature and the Planck constant:

$$c = \frac{2}{\hbar}. \tag{1.12}$$

Physical interpretations of these propositions are given in [8].

1.4 Kähler bundle and Kähler algebra

We review the functional representation of C*-algebra by [9]. We start from their geometric characterization of the set of pure states of a C*-algebra.

Definition 1.6 *A triplet* $(\mathcal{P}, p, B)$ *is a Kähler bundle if* $\mathcal{P}$ *and* B *are topological spaces,* p *is a continuous surjection from* $\mathcal{P}$ *to* B*, and for each* $b \in B$*, its fiber* $\mathcal{P}_b \equiv p^{-1}(b)$ *is a Kähler manifold with respect to the relative topology.*

We do not assume the local triviality of a Kähler bundle. In this paper, we suppose that any manifold is a (possibly uncountable infinite dimensional) Hilbert manifold [24]. Examples of infinite dimensional Kähler Hilbert manifold is a projective Hilbert space, a Hilbert ball and a Loop groups [21].

A triplet $(\mathcal{P}, p, B)$ is a *uniform Kähler bundle* if the topology of $\mathcal{P}$ is a uniform topology [5]. A triplet $(\mathcal{P}, p, B)$ is a *projective Kähler bundle* if each fiber is a projective Hilbert space. A triplet $(\mathcal{P}, p, B)$ is a *hyperbolic Kähler bundle* if each fiber is a Hilbert ball. A triplet $(\mathcal{P}, p, B)$ is a *regular state form* if each fiber is a projective Hilbert space or a Hilbert ball.

Definition 1.7 *Let* $(\mathcal{P}, p, B)$ *be a Kähler bundle. A function* $f \in C^\infty(\mathcal{P})$ *is a Kähler function if* $D^2 f = 0$ *and* $\bar{D}^2 f = 0$ *where* D *and* $\bar{D}$ *are the holomorphic and antiholomorphic part of fiberwise covariant derivative respectively. We write* $\mathcal{K}(\mathcal{P})$ *the set of all Kähler functions on* $(\mathcal{P}, p, B)$*.*

From Proposition 1.5, the following holds.

Theorem 1.8 *Let* $(\mathcal{P}, p, B)$ *be a regular state form. Then the set* $\mathcal{K}(\mathcal{P})$ *of all Kähler functions is a* $*$*-algebra with respect to the complex conjugation* $f^* \equiv \bar{f}$ *and the* $*$*-product defined by*

$$(f * l)(x) \equiv f(x) \cdot l(x) + 2\lambda_x \cdot \partial_x f(\mathrm{grad}_x l) \quad (x \in \mathcal{P},\ f, l \in \mathcal{K}(\mathcal{P})) \tag{1.13}$$

where $\partial_x f$ *is the holomorphic differential of* f*,* $\mathrm{grad}_x l$ *is the holomorphic part of gradient of* l *with respect to the fiberwise Kähler metric and* λ_x *is the inverse of the holomorphic sectional curvature of the Kähler manifold of the fiber at* x*.*

In Theorem 1.8, λ_x is a non zero constant on $p^{-1}(b)$ for $b \in B$. By using these preparation, we show a geometric characterization of the set of pure states and a functional representation of noncommutative C*-algebras.

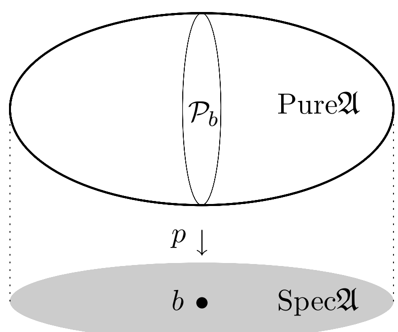
Theorem 1.9 (i) *For a unital C^*-algebra $\mathfrak{A}$, the set $\mathcal{P} \equiv \mathrm{Pure}\mathfrak{A}$ of all pure states of $\mathfrak{A}$ and the spectrum B of $\mathfrak{A}$ (that is, the set $\mathrm{Spec}\mathfrak{A}$ of all unitary equivalence classes of irreducible representations of $\mathfrak{A}$), the natural projection p from $\mathcal{P}$ onto B is a uniform projective Kähler bundle (=UPKB).*

(ii) *The set of all uniform continuous Kähler functions on a UPKB $(\mathcal{P}, p, B)$ in (i) is a unital C^*-algebra and it is $*$-isomorphic to $\mathfrak{A}$.*

(iii) *For two unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, their UPKB's are isomorphic if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-isomorphic.*

(iv) *If $\mathfrak{A}$ is commutative, then the correspondence in (ii) is the Gel'fand representation of commutative C^*-algebras.*

We illustrate the uniform Kähler bundle of a C*-algebra $\mathfrak{A}$ as follows:



By Theorem 1.9, the category of C*-algebras is embedded to the category of Kähler bundles.

For each uniform continuous Kähler function f over the Kähler bundle related a C*-algebra $\mathfrak{A}$, there exists $A \in \mathfrak{A}$ such that

$$f(\rho) = \rho(A) \quad (\rho \in \mathcal{P}).$$

The norm of a function f is given by $\|f\| \equiv \sup_{\rho\in\mathcal{P}} |(\bar{f} * f)(\rho)|^{1/2}$. We can regard a C*-algebra as a special Kähler algebra, that is, the uniform continuous Kähler algebra over a uniform projective Kähler bundle. Furthermore, the Kähler bundle can be regard as the geometric aspects of the noncommutative algebra such the C*-algebra. We showed applications of Theorem 1.9 in [14, 15]. For the other characterization of the state space of a C*-algebra, see [2].

On the other hand, Theorem 1.8 holds not only projective Kähler bundles but also hyperbolic Kähler bundles. Therefore we have a natural question as follows.

Problem 1.10 *What is the Kähler algebra on a hyperbolic Kähler bundle? How is it similar to a C^*-algebra?*

1.5 Kähler geometry of the Hilbert ball

We introduce the Kähler geometry of the Hilbert ball $B_{\mathcal{H}}$ in (1.1) in order to consider the quantum mechanics and operator algebra on $B_{\mathcal{H}}$. Almost statements are shown without difficulty by the analogy of finite dimensional case. Hence their proofs are shown in Appendix A.

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\cdot|\cdot\rangle$ and let $\overline{\mathcal{H}}$ be the conjugate Hilbert space of $\mathcal{H}$. For $v \in \mathcal{H}$, we define $\bar{v} \in \overline{\mathcal{H}}$ by $\bar{v}(w) \equiv \langle v|w\rangle$ for $w \in \mathcal{H}$. Then $B_{\mathcal{H}}$ in (1.1) is an open subset of $\mathcal{H}$. The space $\mathcal{H}$ is a linear Hilbert manifold with the modeled space $\mathcal{H}$. Hence $B_{\mathcal{H}}$ is a sub Hilbert manifold of $\mathcal{H}$. The local coordinate of $B_{\mathcal{H}}$ is the only one which is induced by the inclusion map of $B_{\mathcal{H}}$ into $\mathcal{H}$. Therefore its tangent space $T_zB_{\mathcal{H}}$ at $z \in B_{\mathcal{H}}$ is uniquely defined by $T_zB_{\mathcal{H}} = \mathcal{H}$. We treat $T_zB_{\mathcal{H}}$ and $\overline{T_zB_{\mathcal{H}}}$ as subspaces of $\mathcal{T}_zB_{\mathcal{H}}$ and identify $u \in T_zB_{\mathcal{H}}$, $\bar{v} \in \overline{T_zB_{\mathcal{H}}}$ with $(u,0)$, $(0,\bar{v}) \in \mathcal{T}_zB_{\mathcal{H}}$, respectively. Define the complex structure J on $B_{\mathcal{H}}$ by

$$J_z : \mathcal{T}_zB_{\mathcal{H}} \to \mathcal{T}_zB_{\mathcal{H}}; \quad J_z(u,\bar{v}) \equiv \sqrt{-1}(u,-\bar{v})$$

for $(u,\bar{v}) \in \mathcal{T}_zB_{\mathcal{H}}$ at $z \in B_{\mathcal{H}}$.

Define the Bergman type metric g of $B_{\mathcal{H}}$ on $\mathcal{T}_zB_{\mathcal{H}}$ by

$$g_z((u,\bar{v}),(u^{'},\bar{v}')) \equiv k_z(\langle v|u^{'}\rangle + \langle v^{'}|u\rangle) + k_z^2(\langle v|z\rangle\langle z|u^{'}\rangle + \langle v^{'}|z\rangle\langle z|u\rangle) \quad (1.14)$$

for $(u,\bar{v}),(u^{'},\bar{v}') \in \mathcal{T}_zB_{\mathcal{H}}$ where $k_z \equiv 1/(1-\|z\|^2)$. Then g is invariant under J and symmetric. We write $g_z(u,\bar{v}) = g_z((u,0),(0,\bar{v}))$. By this metric, $(B_{\mathcal{H}},g)$ is a Kähler manifold of constant holomorphic sectional curvature -2. For this metric g, define the Kähler form ω on $B_{\mathcal{H}}$ by (1.2).

We show isometries of the Kähler manifold $B_{\mathcal{H}}$. For this purpose, we define several notations. Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\cdot|\cdot\rangle$. We write the norm $\|\psi\| \equiv \sqrt{\langle\psi|\psi\rangle}$ of $\psi \in \mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and let $\bar{\mathcal{H}}$ be the conjugate Hilbert space of $\mathcal{H}$. Define the new Hilbert space $\widetilde{\mathcal{H}} \equiv \mathcal{H} \oplus \mathbf{C}$. We identify $\mathcal{H}$ as a subspace of $\tilde{\mathcal{H}}$. Then any element of $\mathcal{L}(\tilde{\mathcal{H}})$ is written as follows:

$$\begin{pmatrix} A & x \\ \bar{y} & a \end{pmatrix} \quad (1.15)$$

for $A \in \mathcal{L}(\mathcal{H})$, $a \in \mathbf{C}$ and $x, y \in \mathcal{H}$ where $a \in \mathcal{L}(\mathbf{C}, \mathbf{C})$ is defined by $a \cdot c \equiv ac$ for $c \in \mathbf{C}$, $x \in \mathcal{L}(\mathbf{C}, \mathcal{H})$ is defined by $x(c) = c \cdot x$ and $\bar{y} \in \bar{\mathcal{H}}$ is defined by $\bar{y}(\xi) \equiv \langle y|\xi\rangle$ for each $\xi \in \mathcal{H}$.

Define the *inhomogeneous unitary group* $U_1(\mathcal{H})$ by the subset of bounded linear operators T on $\widetilde{\mathcal{H}}$ satisfying

$$T^*\varepsilon T = \varepsilon$$

where ε is the matrix defined on $\widetilde{\mathcal{H}}$ by

$$\varepsilon \equiv \begin{pmatrix} -I_1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.16}$$

and I_1 is the identity operator on $\mathcal{H}$. Then any element in $U_1(\mathcal{H})$ with the form (1.15) satisfies the following conditions:

$$A^*A - \|y\|^2 \cdot E_y = I_1, \quad \|x\|^2 - |a|^2 = -1, \quad A^*x = ay \tag{1.17}$$

where E_y is the projection from $\mathcal{H}$ to the subspace $\mathbf{C}y$. The action $U_1(\mathcal{H})$ on $B_{\mathcal{H}}$ is given by a *generalized linear fractional transformation* (or a *generalized Möbius transformation*)

$$\phi_T(z) \equiv \frac{Az + x}{\langle y|z\rangle + a} \quad (z \in B_{\mathcal{H}}) \tag{1.18}$$

for $T \in U_1(\mathcal{H})$ with the form (1.15). We see that the group $U_1(\mathcal{H})$ acts on $B_{\mathcal{H}}$ transitively.

Theorem 1.11 *For $u, v \in B_{\mathcal{H}}$, the distance $d(u, v)$ between u and v is given by*

$$d(u, v) = \frac{1}{2} \log \frac{1 - p + \sqrt{\|u - v\|^2 - q^2}}{1 - p - \sqrt{\|u - v\|^2 - q^2}}$$

where $p \equiv \mathrm{Re}\langle u|v\rangle$ and $q \equiv \sqrt{\|u\|^2\|v\|^2 - p^2}$.

The distance d in Theorem 1.11 is called the *Poincaré distance* [17].

Corollary 1.12 *For $u, v \in B_{\mathcal{H}}$, the distance $d(u, v)$ between u and v is given by*

$$\tanh d(u, v) = \frac{\sqrt{\|u - v\|^2 - \|u\|^2\|v\|^2 + (\mathrm{Re}\langle u|v\rangle)^2}}{1 - \mathrm{Re}\langle u|v\rangle}$$

where the hyperbolic tangent $\tanh x$ *is defined by*

$$\tanh x \equiv (e^x - e^{-x})(e^x + e^{-x})^{-1} \quad (x \in \mathbf{R}).$$

Especially, $\mathrm{tanh} d(u, 0) = \|u\|$.

1.6 Main theorem

We show our main statements here. Let $B_{\mathcal{H}}$ be as in (1.1).

Theorem 1.13 (i) *A symmetry of the system on $B_{\mathcal{H}}$ is given by a generalized linear fractional transformation in (1.18) or the mirror transformation*

$$F_W z \equiv E_W z - E_{W^\perp} z$$

where W is a closed real linear subspace of $\mathcal{H}$, E_W is the range projection of W from $\mathcal{H}$ and $W^\perp$ is the orthogonal complementary subspace of W with respect to the real inner product $\mathrm{Re}\langle\cdot|\cdot\rangle$ *of $\mathcal{H}$.*

(ii) *For two states $\phi, \psi \in B_{\mathcal{H}}$, the transition probability $T(\phi, \psi)$ between ϕ and ψ is given by*

$$T(\phi,\psi) = \frac{1}{2}\log\frac{1-p+\sqrt{\|\phi-\psi\|^2-q^2}}{1-p-\sqrt{\|\phi-\psi\|^2-q^2}}$$

where $p \equiv \mathrm{Re}\langle\phi|\psi\rangle$ and $q \equiv \sqrt{\|\phi\|^2\|\psi\|^2 - p^2}$.

Proof. (i) From Definition 1.2 (i) and § 1.5, the statement holds.
(ii) From Definition 1.2 (ii) and Theorem 1.11, the statement holds. □

By Theorem 1.13 (ii), we know that the transition probability in $B_{\mathcal{H}}$ depends on not only $|\langle\phi|\psi\rangle|$ but also the difference of phase factors and lengths of two states.

Theorem 1.14 *In $B_{\mathcal{H}}$, the set of observables is given as the set of smooth functions over $B_{\mathcal{H}}$ with the form*

$$f_C(z) \equiv \frac{\langle\hat{z}|C\hat{z}\rangle}{1-\|z\|^2} \tag{1.19}$$

where $C \in \mathcal{L}(\widetilde{\mathcal{H}})$ and $\hat{z} \equiv (z,1) \in \widetilde{\mathcal{H}}$ for $z \in B_{\mathcal{H}}$. The commutator of functions is given as follows:

$$f_C * f_{C'} - f_{C'} * f_C = -\sqrt{-1}\{f_C, f_{C'}\}$$

where $\{\cdot,\cdot\}$ in the R.H.S. is the Poisson bracket of $B_{\mathcal{H}}$.

We summarize our result about the trial quantum system on $B_{\mathcal{H}}$ here. The case of negative constant holomorphic curvature means that the Planck constant in the system is *negative* by (1.12):

$$\hbar < 0.$$

Hence this makes no sense in the usual physics. Furthermore we do not know the affirmative experimental evidence for this system. However, a physical constant often takes a unusual value in quantum field theory in order to construct a new theory, for example, complex mass, extra dimension and negative energy. Therefore our thought experiment may be useful in the future even if it is only a purely mathematical result.

By Theorem 1.8, we know that the set $\mathcal{K}(B_{\mathcal{H}})$ of all Kähler functions on the Hilbert ball $B_{\mathcal{H}}$ is a $*$-algebra with respect to the $*$-product

$$f * l = f \cdot l - \partial f(\mathrm{grad} l) \quad (f, l \in \mathcal{K}(B_{\mathcal{H}})) \tag{1.20}$$

and the complex conjugation since $B_{\mathcal{H}}$ has the constant holomorphic sectional curvature -2. We determine the structure of $\mathcal{K}(B_{\mathcal{H}})$. We define the new product $*_\varepsilon$ on the algebra $\mathcal{L}(\widetilde{\mathcal{H}})$ of all bounded linear operators on $\widetilde{\mathcal{H}}$ by

$$A *_\varepsilon B \equiv A\varepsilon B \quad (A, B \in \mathcal{L}(\widetilde{\mathcal{H}})).$$

Then $(\mathcal{L}(\widetilde{\mathcal{H}}), *_\varepsilon)$ is a Banach $*$-algebra with the unit ε and the (ordinary) operator norm $\|\cdot\|$. Then the following holds:

$$\|A^* *_\varepsilon I *_\varepsilon A\| = \|A\|^2 \quad (A \in \mathcal{L}(\widetilde{\mathcal{H}})).$$

Define the new norm $\|\cdot\|_b$ of $\mathcal{K}(B_{\mathcal{H}})$ by

$$\|f\|_b = \sup_{(z,\lambda)\in B_{\mathcal{H}}\times \mathbf{R}_+} \left| t_{\lambda^2}(z)^{-1} \cdot (t_\lambda * \bar{f} * h * f * t_\lambda)(z)\right|^{1/2} \quad (f \in \mathcal{K}(B_{\mathcal{H}}))$$

where $\mathbf{R}_+ \equiv \{\lambda \in \mathbf{R} : \lambda \geq 0\}$ and

$$t_\lambda(z) \equiv \frac{\lambda + \|z\|^2}{1 - \|z\|^2}, \quad h(z) \equiv t_1(z) = t_\lambda(z)|_{\lambda=1} \quad (\lambda \geq 0,\ z \in B_{\mathcal{H}}).$$

On these preparation, we state the theorem of structure of the Kähler algebra $\mathcal{K}(B_{\mathcal{H}})$ on $B_{\mathcal{H}}$ in Definition 1.7.

Theorem 1.15 (i) *The triplet $(\mathcal{K}(B_{\mathcal{H}}), *, \|\cdot\|_b)$ is a unital Banach $*$-algebra.*

(ii) $\mathcal{K}(B_{\mathcal{H}}) \cong (\mathcal{L}(\widetilde{\mathcal{H}}), *_{\varepsilon})$ *as a Banach $*$-algebra.*

We consider $(\mathcal{K}(B_{\mathcal{H}}), *, \|\cdot\|_b)$ as a deformation algebra in the theory of deformation quantization [4, 10, 18]. Theorem 1.15 shows that a deformation algebra is isomorphic to a Banach operator algebra with a special operator product. In general, a deformation algebra of a Poisson manifold is neither related to a known algebra except the Moyal algebra [22], nor a Banach algebra. Hence the statement in Theorem 1.15 is very rare. Furthermore the deformation parameter of a deformation algebra takes an indeterminate element in general. On the other hand, the deformation parameter of $\mathcal{K}(B_{\mathcal{H}})$ is uniquely determined by the holomorphic sectional curvature of $B_{\mathcal{H}}$ by (1.11). In this sense, the Kähler structure of $B_{\mathcal{H}}$ is represented by the $*$-product. This is a new example of correspondence between geometry and algebra.

In § 2, we show the proofs of statements in § 1.6. In § 3, we show examples of our results for $B_{\mathcal{H}}$ when $\dim\mathcal{H} = 1$. In § 4, we discuss our results.

2 Proofs of theorems

We prove main theorems in § 1.6.

Lemma 2.1 *The holomorphic part of its Levi-Civita connection on $B_{\mathcal{H}}$ is given by*

$$(D_XY)_z = k_z(\langle z|X\rangle Y + \langle z|Y\rangle X) \quad (z \in B_{\mathcal{H}},\ X, Y \in T_zB_{\mathcal{H}})$$

where $k_z = 1/(1 - \|z\|^2)$.

Proof. We check $D_Xg_z(u, \bar{v}) = 0$ for $X, u \in T_zB_{\mathcal{H}}$ and $\bar{v} \in \overline{T_zB_{\mathcal{H}}}$ for any $z \in B_{\mathcal{H}}$.

$$\begin{aligned} Xg_z(u,\bar{v}) =\ & \partial_z(g_z(u,\bar{v}))(X) \\ =\ & k_z^2\langle v|u\rangle\langle z|X\rangle + 2k_z^3\langle v|z\rangle\langle z|u\rangle\langle z|X\rangle + k_z^2\langle v|X\rangle\langle z|u\rangle, \end{aligned}$$

$$\begin{aligned} g_z(D_Xu,\bar{v}) =\ & k_z\langle v|D_Xu\rangle + k_z^2\langle v|z\rangle\langle z|D_Xu\rangle \\ =\ & k_z^2(\langle v|u\rangle\langle z|Z|\rangle + \langle v|X\rangle\langle z|u\rangle) + 2k_z^3\langle v|z\rangle\langle z|u\rangle\langle z|X\rangle, \end{aligned}$$

$$g_z(u, D_X\bar{v}) =\ 0.$$

Hence $(D_Xg_z)(u,\bar{v}) = Xg_z(u,\bar{v}) - g_z(D_Xu,\bar{v}) - g_z(u, D_X\bar{v}) = 0$. By uniqueness of the Levi-Civita connection, the statement holds. $\square$

Let grad$f$ and sgrad$f$ be as in (1.3). In [3, 9], the notation $G\partial f =$ grad$f$ and $I\partial f =$ sgrad$f$ are used by defining linear mappings G and I on the set of vector fields on a Kähler manifold M. For $M = B_{\mathcal{H}}$, we show some formulae.

Lemma 2.2 *Let $k_z \equiv 1/(1 - \|z\|^2)$. For $z \in B_{\mathcal{H}}$ and $f \in C^\infty(B_{\mathcal{H}})$, the following holds.*

(i) $\mathrm{grad}_z f = k_z^{-1}\{(\bar{\partial}_z f)^* - \langle z|(\bar{\partial}_z f)^*\rangle z\}$,

(ii) $\mathrm{sgrad}_z f = -\sqrt{-1}k_z^{-1}\{(\bar{\partial}_z f)^* - \langle z|(\bar{\partial}_z f)^*\rangle z\}$

where $(\bar{\partial}_z f)^*$ is the vector satisfying $\langle v|(\bar{\partial}_z f)^*\rangle = \bar{\partial}_z f(\bar{v})$ for any vector $v \in \mathcal{H}$.

Lemma 2.2 is proved in Appendix A.

We prove Theorem 1.15 step-by-step.

Lemma 2.3 *For $f \in C^\infty(B_{\mathcal{H}})$, $D^2 f = 0$ if and only if $\partial^2(k^{-1}\cdot f) = 0$ where $k \in C^\infty(B_{\mathcal{H}})$ is defined by $k(z) = (1 - \|z\|^2)^{-1}$ for $z \in B_{\mathcal{H}}$.*

Proof. By Lemma 2.1, for $X, Y \in T_z B_{\mathcal{H}}$, $(D_z^2 f)(X, Y) = 0$ if and only if

$$\begin{aligned}(\partial_z^2 f)(X,Y) &= k(z)\cdot(\partial_z f)(\langle z|X\rangle Y + \langle z|Y\rangle X)\\ &= k(z)\cdot\{\partial_z^2((1-k^{-1})\cdot f)(X,Y) - \|z\|^2\cdot\partial_z^2 f(X,Y)\}.\end{aligned}$$

By multiplying $k(z)^{-1}$ at both sides,

$$(1-\|z\|^2)(\partial_z^2 f)(X,Y) = \partial_z^2((1-k^{-1})\cdot f)(X,Y) - \|z\|^2\cdot\partial_z^2 f(X,Y).$$

It is equivalent to $\partial_z^2(k^{-1}\cdot f)(X,Y) = 0$. □

Lemma 2.4 *If $l \in C^\infty(B_{\mathcal{H}})$ satisfies both $\partial^2 l = 0$ and $\bar{\partial}^2 l = 0$, then there exist $a \in \mathbf{C}$, $A \in \mathcal{L}(\mathcal{H})$ and $u, v \in \mathcal{H}$ such that*

$$l(z) = \langle z|Az\rangle + \langle u|z\rangle + \langle z|v\rangle + a. \tag{2.1}$$

Proof. By assumption of $\partial^2 l = 0$, there exists $u_0 \in \mathcal{H}$ such that

$$(\partial_z l)(X) = \langle u_0|X\rangle \quad (z \in B_{\mathcal{H}}) \tag{2.2}$$

because $(\partial_z l)(X)$ is bounded linear with respect to X. From this, for $\bar{\partial}^2 l = 0$, we see that there exists $v \in \mathcal{H}$ such that $(\bar{\partial}_z l)(\bar{X}) = \langle X|v\rangle$ for $z \in B_{\mathcal{H}}$. From (2.2), there exists $c_0 \in \mathbf{C}$ such that

$$l(z) = \langle u_0|z\rangle + c_0. \tag{2.3}$$

Consider $u_0 = u_0(\bar{z})$ and $c_0 = c_0(\bar{z})$. Then

$$(\bar{\partial}_z l)(\bar{X}) = \bar{\partial}_z\{\langle u_0(z)|z\rangle + c_0(\bar{z})\}(\bar{X}) = \langle \partial_z u_0(X)|z\rangle + \bar{\partial}_z c_0(\bar{X}).$$

Because $(\partial_z u_0)(X)$ and $\bar{\partial}_z c_0(\bar{X})$ are bounded linear with respect to X and $\bar{X}$, respectively, there exist $B \in \mathcal{L}(\mathcal{H})$ and $v \in \mathcal{H}$ such that $\partial_z u_0(X) = BX$ and $\bar{\partial}_z c_0(\bar{X}) = \langle X|v\rangle$. Therefore, there exist $u \in \mathcal{H}$ and $a \in \mathbf{C}$ such that

$$u_0(z) = Bz + u, \quad c_0(z) = \langle z|v\rangle + a. \tag{2.4}$$

By inserting (2.4) into (2.3) and let $A \equiv B^*$, we obtain (2.1). $\square$

For $z \in B_{\mathcal{H}}$, define $\hat{z} \equiv (z, 1) \in \tilde{\mathcal{H}}$.

Definition 2.5 *For $C \in \mathcal{L}(\tilde{\mathcal{H}})$ with the form (1.15), define the function f_C on $B_{\mathcal{H}}$ by*

$$f_C(z) \equiv \frac{a + \langle y|z\rangle + \langle z|x\rangle + \langle z|Az\rangle}{1 - \|z\|^2}.$$

We see that $f_C(z) = k(z) \cdot \langle \hat{z}|C\hat{z}\rangle$ for each $z \in B_{\mathcal{H}}$. We conclude the next corollary.

Corollary 2.6 *If $f \in C^\infty(B_{\mathcal{H}}) \cap \mathcal{K}(B_{\mathcal{H}})$, then there exists $C \in \mathcal{L}(\tilde{\mathcal{H}})$ such that $f = f_C$.*

Proposition 2.7 *For $C, C^{'} \in \mathcal{L}(\tilde{\mathcal{H}})$, $f_C * f_{C'} = f_{C\varepsilon C'}$.*

Proof. We calculate the noncommutative part $\partial_z f_C(\mathrm{grad}_z f_{C'})$ of the $*$-product $f_C * f_{C'}$. By using (A.6) and (A.7) in Lemma A.2,

$$\begin{aligned}
\partial_z f_C(\mathrm{grad}_z f_{C'}) &= \langle (\partial_z f_C)^* | \mathrm{grad}_z f_{C'}\rangle \\
&= \langle \{k_z E_1 C^* \hat{z} + k_z^2 \overline{\langle \hat{z}|C^*\hat{z}\rangle} z\} | \{E_1 C^{'} \hat{z} + \langle e_2|C^{'}\hat{z}\rangle z\}\rangle \\
&= k_z \langle E_1 C^* \hat{z}|C^{'}\hat{z}\rangle + k_z^2 \overline{\langle \hat{z}|C^*\hat{z}\rangle}\langle z|E_1 C^{'}\hat{z}\rangle \\
&\quad + k_z\langle E_1 C^*\hat{z}|z\rangle\langle e_2|C^{'}\hat{z}\rangle + k_z^2\langle \hat{z}|C\hat{z}\rangle\langle e_2|C^{'}\hat{z}\rangle\|z\|^2 \\
&= f_C(z) f_{C'}(z) - k_z^2\langle \hat{z}|C\hat{z}\rangle\langle e_2|C^{'}\hat{z}\rangle \\
&\quad + k_z\langle E_1 C^*\hat{z}|C^{'}\hat{z}\rangle + k_z\langle \hat{z}|Cz\rangle\langle e_2|C^{'}\hat{z}\rangle + k_z^2\langle \hat{z}|C\hat{z}\rangle\langle e_2|C^{'}\hat{z}\rangle\|z\|^2 \\
&= f_C(z) f_{C'}(z) - k_z\langle \hat{z}|Ce_2\rangle\langle e_2|C^{'}\hat{z}\rangle + k_z\langle E_1 C^*\hat{z}|C^{'}\hat{z}\rangle.
\end{aligned}$$

From this, we obtain that

$$\partial_z f_C(\mathrm{grad}_z f_{C'}) = f_C(z) f_{C'}(z) - k_z\langle \hat{z}|C\{\langle e_2|C^{'}\hat{z}\rangle e_2 - E_1 C^{'}\hat{z}\}\rangle. \tag{2.5}$$

Since

$$\langle e_2|C'\hat{z}\rangle e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} C'\hat{z}, \qquad E_1 C'\hat{z} = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} C'\hat{z},$$

the R.H.S. of (2.5) is $f_C(z)f_{C'}(z) - k_z\langle\hat{z}|C\varepsilon C'\hat{z}\rangle = f_C(z)f_{C'}(z) - f_{C\varepsilon C'}(z)$. Therefore,

$$(f_C * f_{C'})(z) = f_C(z)f_{C'}(z) - \partial_z f_C(\mathrm{grad}_z f_{C'}) = f_{C\varepsilon C'}(z).$$

Hence the statement holds. $\square$

Proof of Theorem 1.15. In the first step, we show that $(\mathcal{K}(B_{\mathcal{H}}), *)$ and $(\mathcal{L}(\widetilde{\mathcal{H}}), *_\varepsilon)$ are isomorphic as a $*$-algebra. By Definition 2.5, the mapping f from $\mathcal{L}(\widetilde{\mathcal{H}})$ to $\mathcal{K}(B_{\mathcal{H}})$, $C \mapsto f_C$, is defined. By Corollary 2.6, f is surjective. If $f_C = 0$ for $C = \begin{pmatrix} A & \bar{u} \\ v & a \end{pmatrix} \in \mathcal{L}(\widetilde{\mathcal{H}})$, then by comparing degree of $z \in B_{\mathcal{H}}$ in f_C, $\langle z|Az\rangle = 0$, $\langle u|z\rangle + \langle z|v\rangle = 0$, $a = 0$ for any $z \in B_{\mathcal{H}}$. Hence $A = 0$. Since $\langle u|z\rangle$ is affine and $\langle z|v\rangle$ is conjugate affine with respect to z, we obtain $u = v = 0$. Therefore $C = 0$. Hence f is injective. Clearly, f is linear and $f_{C^*} = \bar{f}_C$. By Proposition 2.7, f is a $*$-isomorphism.

By definition of the norm $\|\cdot\|_b$,

$$\|f_C\|_b^2 = \sup_{\lambda \ge 0,\, z \in B_{\mathcal{H}}} \left| t_{\lambda^2}(z)^{-1} \cdot (t_\lambda * \bar{f}_C * h * f_C * t_\lambda)(z) \right|.$$

We can write $t_\lambda = f_{T_\lambda}$ by $T_\lambda \equiv \begin{pmatrix} I_1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $h = f_I$. From Proposition 2.7 and (1.19),

$$\|f_C\|_b^2 = \sup_{(z,\lambda) \in B_{\mathcal{H}} \times \mathbf{R}_+} \left| \left(\frac{\lambda^2 + \|z\|^2}{1 - \|z\|^2} \right)^{-1} \cdot (f_{T_\lambda *_\varepsilon C^* *_\varepsilon I *_\varepsilon C *_\varepsilon T_\lambda})(z) \right|.$$

Since

$$\left(\frac{\lambda^2 + \|z\|^2}{1 - \|z\|^2} \right)^{-1} \cdot (f_{T_\lambda *_\varepsilon C^* *_\varepsilon I *_\varepsilon C *_\varepsilon T_\lambda})(z) = \frac{\langle(-z,\lambda)|C^*C(-z,\lambda)\rangle}{\lambda^2 + \|z\|^2} = \frac{\|C(-z,\lambda)\|^2}{\|(-z,\lambda)\|^2},$$

we obtain $\|f_C\|_b^2 = \|C\|^2$. Hence f is an isometry from the Banach space $(\mathcal{L}(\widetilde{\mathcal{H}}), \|\cdot\|)$ to $(\mathcal{K}(B_{\mathcal{H}}), \|\cdot\|_b)$. Therefore $(\mathcal{L}(\widetilde{\mathcal{H}}), *, \|\cdot\|)$ is a Banach $*$-algebra with the unit $f_J \equiv 1$ because $(\mathcal{L}(\widetilde{\mathcal{H}}), *_\varepsilon, \|\cdot\|)$ is a Banach $*$-algebra. We finish to show (i). The statement (ii) is clear from the proof of (i). $\square$

Proof of Theorem 1.14. An observable in the system is given by a function f on $B_{\mathcal{H}}$ such that $D^2 f = 0$ and $\bar{D}^2 f = 0$ where $D, \bar{D}$ are the holomorphic, the antiholomorphic part of covariant derivative of $B_{\mathcal{H}}$ respectively. Therefore it is calculated in Theorem 1.15. Later equation follows from Theorem 1.8. □

Remark 2.8 As same as the projective Hilbert space case, we can consider a manifold domain [7] of $B_{\mathcal{H}}$ for a unbounded selfadjoint operator H with dense domain $\mathcal{D} \subset \mathcal{H}$. For such $(H, \mathcal{D})$, we can make $\mathcal{D}$ as a Hilbert space with the inner product $\langle \cdot | \cdot \rangle_H$

$$\langle x|y \rangle_H \equiv \langle x|y \rangle + \langle Hx|Hy \rangle \quad (x, y \in \mathcal{D}).$$

Let $B_{\mathcal{D}} \equiv B_{\mathcal{H}} \cap \mathcal{D}$ is also a Kähler (Hilbert) manifold. The inclusion mapping ι of $B_{\mathcal{D}}$ into $B_{\mathcal{H}}$ is an isometry and the range of its differential $d\iota$ is dense in each tangent space of $B_{\mathcal{H}}$.

Remark 2.9 The new algebra $(\mathcal{L}(\widetilde{\mathcal{H}}), *_\varepsilon)$ and the algebra $(\mathcal{L}(\widetilde{\mathcal{H}}), \cdot)$ with the (ordinary) operator product $\cdot$ are isomorphic as a unital Banach algebra by the mapping

$$A \mapsto \varepsilon A.$$

However, $(\mathcal{L}(\widetilde{\mathcal{H}}), *_\varepsilon)$ and $(\mathcal{L}(\widetilde{\mathcal{H}}), \cdot)$ are not $*$-isomorphic because $(\mathcal{L}(\widetilde{\mathcal{H}}), *_\varepsilon)$ does not satisfy the C*-condition. We explain this as follows: Let $v \in \mathcal{H}$ be a unit vector and let E be the one dimensional projection from $\mathcal{H}$ to $\mathbf{C}v$. Then for an operator

$$A \equiv \begin{pmatrix} E & 0 \\ v & 0 \end{pmatrix},$$

the following equations hold:

$$A^* A = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^* \varepsilon A = 0.$$

Hence, $\|A\|^2 = \|A^* A\| \neq \|A^* \varepsilon A\| = \|A^* *_\varepsilon A\|$.

Remark 2.10 If we consider $\mathcal{K}(B_{\mathcal{H}})$ as a functional representation of some kind of operator algebra and $\mathcal{K}(B_{\mathcal{H}})$ has a norm, then we will expect that $\mathcal{K}(B_{\mathcal{H}})$ becomes a norm algebra. But if we treat $\mathcal{K}(B_{\mathcal{H}})$ as a function algebra on $B_{\mathcal{H}}$, then we should consider a norm which is defined on only $B_{\mathcal{H}}$. Therefore the norm $\|\cdot\|_b$ is not suitable for $\mathcal{K}(B_{\mathcal{H}})$.

For example, we define the other norm $\|\cdot\|_d$ by

$$\|f\|_d \equiv \sup_{z\in B_{\mathcal{H}}} |h(z)^{-1}\cdot(\bar{f}*h*f)(z)|^{1/2} \quad (f\in\mathcal{K}(B_{\mathcal{H}})).$$

Then

$$\|f_C\|_d = \|C\|_s \equiv \sup_{z\in B_{\mathcal{H}}} \frac{\|C(z,1)\|}{\|(z,1)\|} \qquad (C\in\mathcal{L}(\widetilde{\mathcal{H}})).$$

Both $(\mathcal{K}(B_{\mathcal{H}}),\|\cdot\|_d)$ and $(\mathcal{L}(\widetilde{\mathcal{H}}),\|\cdot\|_s)$ are Banach spaces and they are $*$-isomorphic as a $*$-algebra and isometric but not Banach $*$-algebras because they don't satisfy the condition

$$\|A*_\varepsilon B\|_s \le \|A\|_s\cdot\|B\|_s \quad (A,B\in\mathcal{L}(\widetilde{\mathcal{H}})).$$

For example, let $v\in\mathcal{H}$, $\|v\|=1$ and let $E:\mathcal{H}\to\mathbf{C}v$ be the one dimensional projection. Define two matrices C and C$^{'}$ on $\widetilde{\mathcal{H}}\equiv\mathcal{H}\oplus\mathbf{C}$ by

$$C\equiv\begin{pmatrix} E & 0\\ 0 & 0\end{pmatrix},\quad C^{'}\equiv\begin{pmatrix} 0 & \bar{v}\\ 0 & 0\end{pmatrix}.$$

Then $\|C*_\varepsilon C^{'}\|_s \not\le \|C\|_s\cdot\|C^{'}\|_s$.

3 1-dimensional case — unit open disc

We consider the quantum mechanics on $B_{\mathcal{H}}$ for the case of $\dim_{\mathbf{C}}\mathcal{H}=1$.

3.1 Geometry and Kähler algebra

In this case, $B_{\mathcal{H}}$ is the unit open disc D in $\mathbf{C}$ defined by

$$D\equiv\{z\in\mathbf{C}:|z|<1\}.$$

The space D is well-known as an example of hyperbolic complex space [12, 17]. We review properties of D. The Kähler metric g on D is given by the *Poincaré metric* defined by

$$g_z(\bar{v},u)=2\bar{v}u/(1-|z|^2)^2 \quad (z\in D,\,\bar{v},u\in\mathbf{C})$$

where g is scaled such that the curvature of (D,g) is -2. For $z_1,z_2\in D$, the geodesic ℓ_1 between z_1 and z_2 is the circular arcs perpendicular to the boundary $|z|=1$ (Fig. 4.1). In particular, if z_1, z_2 and 0 are on the same straight line ℓ_2, then ℓ_2 is the geodesic between z_1 and z_2 (Fig. 4.2).

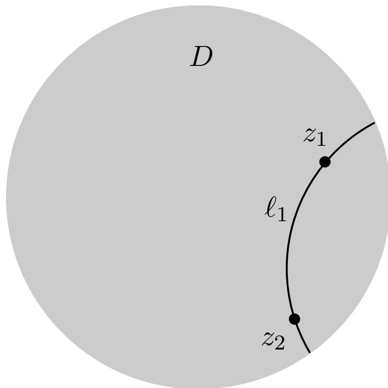


Fig. 4.1

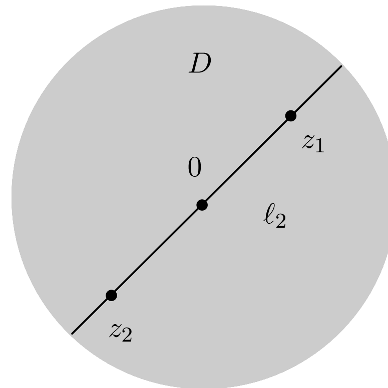


Fig. 4.2

If $\dim_{\mathbf{C}} \mathcal{H} \neq 1$, then the geodesic between z_1 and z_2 in $B_{\mathcal{H}}$ is always transformed to the straight line l through 0 by a suitable isometry of $B_{\mathcal{H}}$. In this case, there exists a 1-dimensional complex subspace $V \subset \mathcal{H}$ containing l such that $\{v \in V : \|v\| < 1\}$ is isometric to D. Hence the above illustration also makes sense in the general case.

The group of all isometries of D is generated by a linear fractional transformation ϕ_X by $X \in M_2(\mathbf{C})$ with the form

$$X = \begin{pmatrix} s & r \\ \bar{r} & \bar{s} \end{pmatrix}, \quad |s|^2 - |r|^2 = 1 \tag{3.1}$$

and the complex conjugation $z \mapsto \bar{z}$.

The Kähler algebra on D is $*$-isomorphic to $(M_2(\mathbf{C}), *_\varepsilon)$ where $*_\varepsilon$ is the product of $M_2(\mathbf{C})$ defined by

$$A *_\varepsilon B \equiv A\varepsilon B \quad (A, B \in M_2(\mathbf{C})) \quad \varepsilon \equiv \mathrm{diag}(-1, 1) \in M_2(\mathbf{C}).$$

3.2 Dynamics of states in D

In the trial quantum system of D, the dynamical law is given by a continuous one parameter group $\{\Phi_t\}_{t\in\mathbf{R}}$ of Kähler isometries on D by Definition 1.1. From (3.1), the Lie algebra $\mathfrak{g}$ of the group of all isometries of D is given as

$$\left\{ \begin{pmatrix} \sqrt{-1}a & b \\ \bar{b} & -\sqrt{-1}a \end{pmatrix} : a \in \mathbf{R},\, b \in \mathbf{C} \right\}.$$

For a given $\{\Phi_t\}_{t\in\mathbf{R}}$, there exists $X \in \mathfrak{g}$ such that $\Phi_t = \phi_{\exp tX}$ for each t. From this, we can concretely describe the time evolution as

$$z(t) \equiv \phi_{\exp tX}(z) \quad (z \in D,\, t \in \mathbf{R}).$$

For $X \equiv \begin{pmatrix} \sqrt{-1}a & b \\ \bar{b} & -\sqrt{-1}a \end{pmatrix} \in \mathfrak{g}$, let $\alpha \equiv |b|^2 - a^2$. If $\alpha > 0$, then

$$z(t) = \frac{(\sqrt{\alpha} + \sqrt{-1}a \tanh \sqrt{\alpha}t)z + b \tanh \sqrt{\alpha}t}{(\bar{b} \tanh \sqrt{\alpha}t)z + \sqrt{\alpha} - \sqrt{-1}a \tanh \sqrt{\alpha}t}.$$

If $\alpha < 0$, then

$$z(t) = \frac{(\sqrt{-\alpha} + \sqrt{-1}a \tan \sqrt{-\alpha}t)z + b \tan \sqrt{-\alpha}t}{(\bar{b} \tan \sqrt{-\alpha}t)z + \sqrt{-\alpha} - \sqrt{-1}a \tan \sqrt{-\alpha}t}.$$

In addition to this condition, if $b = 0$, then

$$z(t) = e^{2\sqrt{-1}at}z.$$

If $\alpha = 0$, then $z(t) = z$ for each t. In § 2.1 of [23], the Schrödinger equation is derived from the assumption that the Hamiltonian is the generator of time evolution [11]. More properly, the Hamiltonian $\times\sqrt{-1}$ in the ordinary quantum mechanics is chosen as the infinitesimal generator of a one-parameter unitary group on the Hilbert space. Therefore an element $H \in \frac{1}{\sqrt{-1}}\mathfrak{g}$ is a candidate for a Hamiltonian operator of the trial quantum mechanics on D.

4 Discussion

We discuss relations between cases of $\mathcal{P}(\mathcal{H})$ and $B_{\mathcal{H}}$. We show the quantum mechanics of the Hilbert ball including that of the projective Hilbert space as a special case.

Assume that $H \in \mathcal{L}(\mathcal{H})$ is selfadjoint and $a \in \mathbf{R}$. Define the operator X on $\widetilde{\mathcal{H}}$ by

$$X \equiv \begin{pmatrix} \sqrt{-1}H & 0 \\ 0 & \sqrt{-1}a \end{pmatrix}.$$

Then $\exp tX \in U_1(\mathcal{H})$ for each $t \in \mathbf{R}$ because it satisfies (1.17). Let $z(t) \equiv \phi_{\exp tX}(z)$ for $z \in B_{\mathcal{H}}$. Then $z(t) = e^{\sqrt{-1}(H-aI)t}z$. If we replace H by $-H + Ia$, then

$$z(t) = e^{-\sqrt{-1}Ht}z.$$

This is the solution of the Schrödinger equation with the Hamiltonian H:

$$Hz(t) = \sqrt{-1}\frac{d}{dt}z(t).$$

From Corollary 1.12, the inner product $\langle\cdot|\cdot\rangle$ of $\mathcal{H}$ is recovered from the Kähler distance d of $B_{\mathcal{H}}$ as follows. If $u, v \in B_{\mathcal{H}}$ satisfy $\|u\| = \|v\|$, then

$$\begin{aligned}\langle u|v\rangle &= 2\sinh^2(\tfrac{d(u,v)}{2}) + \tanh^2 d(u,0)\cosh d(u,v) \\ &\quad +\sqrt{-1}\{2\sinh^2(\tfrac{d(\sqrt{-1}u,v)}{2}) + \tanh^2 d(u,0)\cosh d(\sqrt{-1}u,v)\}.\end{aligned}$$

Appendix

A Proof of lemmata

Lemma A.1 *For $z \in B_{\mathcal{H}}$ and $0 \le t \le 1$, define $\theta(t) \in B_{\mathcal{H}}$ by*

$$\theta(t) \equiv t \cdot z. \tag{A.1}$$

Then θ is the geodesic between 0 and z. The length $L(\theta)$ of θ is given by

$$L(\theta) = \frac{1}{2}\log\frac{1+\|z\|}{1-\|z\|}.$$

Proof. We show $L(\theta) \le L(\psi)$ for any curve ψ between 0 and z. By definition of length of curve

$$L(\psi) = \int_0^1 \sqrt{K(t)}\,dt \tag{A.2}$$

where $K(t) \equiv g_{\psi(t)}(\dot\psi(t), \dot\psi(t))$. Then

$$K(t) = \frac{\|\dot\psi(t)\|^2}{1-\|\psi(t)\|^2} - \frac{|\langle\dot\psi(t)|\psi(t)\rangle|^2}{(1-\|\psi(t)\|^2)^2}.$$

By the Schwartz inequality for the second term, we obtain

$$K(t) \ge \frac{\|\dot\psi(t)\|^2}{1-\|\psi(t)\|^2} - \frac{\|\dot\psi(t)\|^2\|\psi(t)\|^2}{(1-\|\psi(t)\|^2)^2} = \frac{\|\dot\psi(t)\|^2}{(1-\|\psi(t)\|^2)^2}.$$

Hence $K(t) = \frac{\|\dot\psi(t)\|^2}{(1-\|\psi(t)\|^2)^2}$ if and only if there exists a curve $\alpha : [0,1] \to \mathbf{C}$ such that $\dot\psi(t) = \alpha(t)\psi(t)$. The solution of this equation is given by $\psi(t) = C\exp(\int_0^t \alpha(t)dt) \cdot z$ where C is an integral constant. Since the length of geodesic is invariant for reparametrization and the initial condition of ψ is $\phi(0) = 0$, we find $\psi(t) = t \cdot z$. Hence the first statement is verified. Applying $K(t) = \frac{\|z\|^2}{(1-t^2\|z\|^2)^2}$ to (A.2), we obtain the second statement. $\square$

Proof of Theorem 1.11. We calculate the distance on $B_{\mathcal{H}}$ according to *"Exercises and Further Results, G. The Hyperbolic Plane"* in § 1 of [12] and § 2 of [17]. If $u = 0$ or $v = 0$, then it is given by Lemma A.1 and in this case, the statement is true. Assume $u \neq 0$ and $v \neq 0$. If $v = k \cdot u$ for some $k \in \mathbf{R}$, then $u, v, 0$ are on a same geodesic by Lemma A.1. Hence we assume that u and v are $\mathbf{R}$-linearly independent.

Let $e_1 \equiv u/\|u\|$, $x = \mathrm{Re}\langle e_1|v\rangle$, $e_2 \equiv \frac{v - xe_1}{\|v - xe_1\|}$, $y \equiv \mathrm{Re}\langle e_2|v\rangle$. Then $x = p/\|u\|$, $y = q/\|u\|$, $p^2 + q^2 = \|u\|^2\|v\|^2$ and

$$v = xe_1 + ye_2 = (pe_1 + qe_2)/\|u\|.$$

Let $T = \begin{pmatrix} A & m_u u \\ m_u \bar{u} & m_u \end{pmatrix} \in U_1(\mathcal{H})$ where $m_u \equiv (1 - \|u\|^2)^{-1/2}$. Then we see that $\phi_T(0) = u$ for any $A \in \mathcal{L}(\mathcal{H})$ satisfying (1.17) for $x = y = m_u u$ and $a = m_u$.

From Lemma A.1, the distance $d(0, w)$ between 0 and $w \equiv \phi_{T^{-1}}(v)$ is given by

$$d(0, w) = \frac{1}{2} \log \frac{1 + \|w\|}{1 - \|w\|}.$$

Because $\phi_{T^{-1}}$ is also an isometry, $d(u, v) = d(0, w)$. By definition of w and v,

$$w = \{(p/\|u\| - \|u\|)e_1 + (qm_u^{-1}/\|u\|)e_2\}(1 - p)^{-1}.$$

From this, we obtain $\|w\|^2(1 - p)^2 = \|u - v\|^2 - q^2$. Hence the statement holds. □

Proof of Lemma 2.2. (i) Let $W \equiv \mathrm{grad} f$. By definition, $g_z(W_z, \bar{v}) = \bar{\partial}_z f(\bar{v})$. The L.H.S. of this is

$$k_z\langle v|W_z\rangle + k_z^2\langle v|z\rangle\langle z|W_z\rangle = \langle v|(k_z W_z + k_z^2\langle z|W_z\rangle z)\rangle$$

from (1.14). From this,

$$(\bar{\partial}_z f)^* = k_z W_z + k_z^2\langle z|W_z\rangle z. \tag{A.3}$$

Hence

$$\langle z|(\bar{\partial}_z f)^*\rangle = (k_z + k_z^2\|z\|^2)\langle z|W_z\rangle = k_z^2\langle z|W_z\rangle. \tag{A.4}$$

By inserting (A.4) to (A.3), $W_z = k_z^{-1}\{(\bar{\partial}_z f)^* - \langle z|(\bar{\partial}_z f)^*\rangle z\}$.

(ii) Let $W \equiv \mathrm{sgrad} f$. Then $\omega_z(W_z, \bar{u}) = \bar{\partial}_z f(\bar{u})$. The L.H.S. of this is

$$g_z(J_z W_z, \bar{u}) = \sqrt{-1}(k_z\langle u|W_z\rangle + k_z^2\langle u|z\rangle\langle z|W_z\rangle).$$

Hence we obtain that $(\bar{\partial}_z f)^* = \sqrt{-1}\{k_z W_z + k_z^2 \langle z|W_z\rangle z\}$. As same way about (i), we can verify the statement of (ii). $\Box$

Lemma A.2 *Let E_1 be the projection from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$ and $e_2 \equiv (0,1) \in \widetilde{\mathcal{H}}$. Then the following holds:*

$$
\begin{aligned}
(\partial_z f_C)(u) &= k_z\langle \hat{z}|Cu\rangle + k_z^2\langle \hat{z}|C\hat{z}\rangle\langle z|u\rangle, && \text{(A.5)}\\
(\partial_z f_C)^* &= k_z E_1 C^*\hat{z} + k_z^2\langle \hat{z}|C^*\hat{z}\rangle z, && \text{(A.6)}\\
\mathrm{grad}_z f_C &= E_1 C\hat{z} + \langle e_2|C\hat{z}\rangle z && \text{(A.7)}
\end{aligned}
$$

where $(\partial_z f_C)^*$ is the holomorphic tangent vector at z defined by $\langle(\partial_z f_C)^*|X\rangle = (\partial_z f_C)(X)$ for $X \in T_z B_{\mathcal{H}}$.

Proof. We show only the later two parts. From (A.5),

$$
\begin{aligned}
(\partial_z f_C)(u) &= \langle k_z C^*\hat{z}|u\rangle + \langle\, k_z^2 \overline{\langle \hat{z}|C\hat{z}\rangle} z\, |u\rangle\\
&= \langle\{k_z C^*\hat{z} + k_z^2\langle \hat{z}|C^*\hat{z}\rangle z\}|u\rangle\\
&= \langle\{k_z C^*\hat{z} + k_z^2\langle \hat{z}|C^*\hat{z}\rangle z\}|E_1 u\rangle\\
&= \langle\{k_z E_1 C^*\hat{z} + k_z^2\langle \hat{z}|C^*\hat{z}\rangle z\}|u\rangle.
\end{aligned}
$$

By comparing both side of this equation, we obtain (A.6). By inserting (A.6) to Lemma 2.2, we find (A.7). $\Box$

References

[1] M.C. Abbati, R. Cirelli, P. Lanzavecchia and A. Manià, Pure states of general quantum-mechanical systems as Kähler bundles, Nuovo Cimento B 83, 43-60 (1984).

[2] E.M. Alfsen and F.W. Shultz, Geometry of state spaces of operator algebras, Birkhauser Boston, Inc., Boston, MA, 2003.

[3] V.I. Arnold, Mathematical methods of classical mechanics, Springer-Verlag, 1989.

[4] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and quantization, I, II, Ann. Phys. 111 (1978) 61-110, 111-151.

[5] N. Bourbaki, Elements of mathematics, General topology part I, Addison-Wesley Publishing Company, 1966.

[6] R. Cirelli and P. Lanzavecchia, Hamiltonian vector fields in quantum mechanics, Nuovo Cimento B 79 (1983) 271-283.

[7] R. Cirelli, P. Lanzavecchia and A. Manià, Normal pure states of the von Neumann algebra of bounded operators as Kähler manifold, J. Phys. A 16 (1983) 3829-3835.

[8] R. Cirelli, A. Manià and L. Pizzocchero, Quantum mechanics as an infinite dimensional Hamiltonian system with uncertainty structure, Part I;Part II, J. Math. Phys. 31 (1990) 2891-2897, 2898-2903.

[9] ——, A functional representation of non-commutative C*-algebras, Rev. Math. Phys. 6, 5 (1994) 675-697.

[10] B. Fedosov, Deformation quantization and index theory, Mathematical Topics 9, Akademie Verlag, 1966.

[11] H. Goldstein, Classical mechanics, 2nd edit., Addison-Wesley Publishing Company, 1980.

[12] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, 1978.

[13] A. Heslot, Quantum mechanics as a classical theory, Phys. Rev. D 31, 6 (1985) 1341-1348.

[14] K. Kawamura, Infinitesimal Takesaki duality of Hamiltonian vector fields on a symplectic manifold, Rev. Math. Phys. 12, 12 (2000) 1669-1688.

[15] ——, Serre-Swan theorem for non-commutative C*-algebras, J. Geom. Phys. 48 (2003) 275-296.

[16] S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol I, Interscience Publishers, 1969.

[17] S. Kobayashi, Hyperbolic complex spaces, Springer-Verlag, 1998.

[18] M. Kontsevich, Deformation quantization of Poisson manifolds I, Lett. Math. Phys. 66 (2003) 157-216.

[19] N.H. Kuiper, The homotopy type of the unitary group of Hilbert space, Topology, 3 (1965) 19-30.

[20] L.D. Landau and E.M. Lifshitz, Quantum mechanics (Non-relativistic theory), Elsevier Science Ltd., 1977.

[21] A. Pressley and G. Segal, Loop groups, Oxford Science Publications, 1986.

[22] J.E. Moyal, Quantum mechanics as a statistical theory, Proc. Cambridge Phil. Soc. 45 (1949) 99-124.

[23] J.J. Sakurai, Modern quantum mechanics, revised edit., Addison-Wesley Publishing Company, 1994.

[24] J.T. Schwartz, Non linear functional analysis, Gordon and Breach Science Publishers, 1969.

[25] N.M.J. Woodhouse, Geometric quantization, 2nd edit., Oxford Univ. Press, 1992.